arXiv:0810.4909v1 [hep-th] 27 Oct 2008

Wall crossing in local Calabi Yau manifolds

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ABSTRACT: We study the BPS states of a $D6$-brane wrapping the conifold and bound to collections of $D2$ and $D0$ branes. We find that in addition to the complexified Kähler parameter of the rigid $\mathbb{P}^1$ it is necessary to introduce an extra real parameter to describe BPS partition functions and marginal stability walls. The supergravity approach to BPS state-counting gives a simple derivation of results of Szendrői concerning Donaldson-Thomas theory on the noncommutative conifold. This example also illustrates some interesting limitations on the supergravity approach to BPS state-counting and wall-crossing.

Contents

1. Introduction

One of the most effective tools in the trade of counting BPS states of D-branes wrapped on Calabi-Yau manifolds is the approach using multi-centered solutions of the low energy effective supergravity [6, 7]. This approach leads to simple and intuitive derivations of wall-crossing formulae for BPS indices [9]. There has also been a great deal of activity on the closely related subject of wall-crossing formulae for generalized Donaldson-Thomas invariants in the mathematical literature. In this paper we use the supergravity formulae to make contact with one set of mathematical results, due to B. Szendrői [30, 34]. A simple variation of our arguments should reproduce the related results of J. Bryan and B. Young [35]. In the process we highlight some interesting ways in which the supergravity techniques can break down.

Here is a brief overview of the paper: In section 2 we review the definition of the D6/D2/D0 partition function both from the microscopic viewpoint, and from the macroscopic - or supergravity - viewpoint. In section 3 we summarize previous calculations of Donaldson-Thomas theory on the conifold at large Kähler class as well as on a noncommutative resolution of the conifold. In section 4 we describe a method for applying

wall-crossing techniques to local Calabi-Yau manifolds. Curiously, we find it necessary to use an extension of the complexified Kähler moduli space of the local Calabi-Yau by an extra real parameter φ. In section 5 we determine the walls of marginal stability for D6/D2/D0 bound states on the resolved conifold, and compute the D6/D2/D0 partition function for certain ranges of φ.

As a steadfast reader will discover, this paper raises several unanswered questions. Perhaps the most interesting of these are to be found in section 6. There we explore some puzzles associated with crossing a wall of threshold stability.

During the final stage of preparation of this work, we received [22, 23] in which a similar partition function of D6/D2/D0 bound states on the resolved conifold is computed using different techniques.

2. Review of BPS black holes and wall crossing

Consider IIA string theory compactified on a Calabi-Yau 3-fold X. The resulting effective supergravity in $3 + 1$ dimensions has $\mathcal{N} = 2$ supersymmetry. There are $(h_{11}(X) + 1)$ independent $U(1)$ vector fields obtained from the Kaluza-Klein reduction of the RR fields along even harmonic forms. There exist half-BPS black holes in these theories, which are the supergravity description of BPS bound states of D-branes. Roughly speaking these correspond to solutions of Hermitian Yang-Mills equations on holomorphic cycles in X. Such supersymmetric boundary conditions are thought to be equivalent to the "stable" objects in the bounded derived category of X. They are characterized by their charges under the $U(1)$ vector fields. For the purposes of this paper these charges can be thought of as elements of $H^{even}(X; \mathbb{R})$; we adhere to the conventions of [9].

In addition to the ordinary single centered extremal charged black holes, there exist multi-centered BPS solutions with mutually nonlocal constituents [6, 7]. These configurations are (generically) true bound states: the binding energy is negative. Moreover, once the overall center of mass is factored out, the classical moduli space of such supergravity solutions is compact.

In the present work, we will be concerned with a special case of such objects, namely those with a single unit of D6 charge. It is convenient to define the generating function of the index of BPS states with such charges as a formal power series:

$$Z(u, v; t_\infty) := \sum_{N \in \mathbb{Z}, \beta \in H^4(X, \mathbb{Z})} u^N v^\beta \Omega(1 - \beta + N dV; t_\infty). \tag{2.1}$$

Here dV is a generator of $H^6(X; \mathbb{Z})$ and we have turned on chemical potentials v and u for the D2 and D0 charges, respectively. We denote the complexified Kähler modulus by $t = B + iJ$, and t_∞ refers to the boundary conditions at spatial infinity for these moduli. Note that by a shift of B-field we can, without loss of generality, assume that the $D4$ charge is zero.

2.1 Index of BPS states in $\mathcal{N} = 2$ supergravity

The index of BPS states $\Omega(1 - \beta + NdV; t_\infty)$ appearing in (2.1) is a piecewise constant integer function of the asymptotic moduli t_∞. Moreover, Ω can only jump when the

asymptotics of the effective potential on the configuration space change. The only known way that this can occur is when the physical size of a multi-centered solution diverges for some values of the asymptotic Kähler moduli.

The multi-centered BPS black hole solutions found by Denef (see [9, 5] for notation) are encoded by a harmonic function $H : \mathbb{R}^3 \to H^{even}(X, \mathbb{R})$ with poles at the centers of the constituents. The metric is given by

$$ds^2 = -e^{2U}(d\tau + \omega)^2 + e^{-2U}d\vec{x}^2, \qquad (2.2)$$

where the warp factor (and the spatially-dependent Calabi-Yau moduli) are determined by the attractor equation [13, 29]:

$$2e^{-U}\mathrm{Im}(e^{-i\alpha}\Omega_{\mathrm{nrm}}) = -H. \qquad (2.3)$$

Here α is the argument of the central charge of the total (RR gauge theory) charge of the bound state at the background values of the Kähler moduli, $\alpha(\vec{x}) := \arg\left(\sum_i Z(\Gamma_i; t(\vec{x}))\right)$. The normalized periods are given at large volume by $\Omega_{\mathrm{nrm}} = -\sqrt{\frac{3}{4J_\infty^3}}e^{B+iJ}$. The one-form ω is determined by

$$*_3 d\omega = \langle dH, H \rangle \qquad (2.4)$$

where the Hodge star is defined in flat $\mathbb{R}^3$, and we use the symplectic form on $H^{even}(X, \mathbb{R})$ given by mirror symmetry (or, equivalently, by the natural symplectic form on $K^0(X)$.)

For a configuration with n centers of charge Γ_i, the harmonic function is given by

$$H(\vec{x}) = \sum_{i=1}^{n} \frac{\Gamma_i}{|\vec{x} - \vec{x}_i|} - 2\mathrm{Im}(e^{-i\alpha}\Omega_{\mathrm{nrm}})|_{r=\infty}. \qquad (2.5)$$

The asymptotic value of $H(\vec{x})$ for $\vec{x} \to \infty$ will be denoted by H_∞. It is related by the attractor equations to the asymptotic values of the Kähler moduli, t_∞.

For a two centered solution of this type, the distance between the centers can be calculated to be [8]

$$r_{12} = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2\mathrm{Im}(e^{-i\alpha_\infty}Z(\Gamma_1; t_\infty))}. \qquad (2.6)$$

This quantity must be positive for the solution to exist - the resulting condition on t_∞ is called the Denef stability condition. Furthermore, at a wall of marginal stability in the Kähler moduli space, where $Z(\Gamma_1; t_\infty) = \gamma Z(\Gamma_2; t_\infty)$, with $\gamma \in \mathbb{R}_+$, the radius r_{12} diverges. The bound state thus decays as t_∞ crosses such a real codimension one wall.

At a wall of marginal stability corresponding to a decay $\Gamma \to \Gamma_1 + \Gamma_2$, the BPS index will have a discrete jump given by [9]

$$\Delta\Omega(\Gamma, t) = (-1)^{\langle \Gamma_1, \Gamma_2 \rangle - 1} |\langle \Gamma_1, \Gamma_2 \rangle| \, \Omega(\Gamma_1, t_{\mathrm{ms}})\Omega(\Gamma_2, t_{\mathrm{ms}}), \qquad (2.7)$$

when Γ_1 and Γ_2 are primitive, and $t = t_{\mathrm{ms}}$ is the point (assumed generic) where the wall is crossed. In this paper we will also need a generalization to decays where one of the

constituent charges is primitive, but the other is not. This semi-primitive wall crossing formula is also given in [9],

$$\Omega(\Gamma_1;t) + \sum_{N\geq 1} \Delta\Omega(\Gamma_1 + N\Gamma_2;t)q^N = \Omega(\Gamma_1;t)\prod_{k\geq 1}(1-(-1)^{k\langle\Gamma_1,\Gamma_2\rangle}q^k)^{k|\langle\Gamma_1,\Gamma_2\rangle|\Omega(k\Gamma_2;t)}. \quad (2.8)$$

In this paper we will not need the more elaborate Kontsevich-Soibelman wall-crossing formula [18, 14].

It is worth noting that the regions of Denef stability are bounded by walls of marginal stability as well as by walls of *anti* marginal-stability, where

$$\arg\left(Z(\Gamma_1;t_\infty)\right) = -\arg\left(Z(\Gamma_2;t_\infty)\right).$$

Note that in general, the Denef stability condition for a two centered solution,

$$\langle\Gamma_1,\Gamma_2\rangle\mathrm{Im}\left(Z^*(\Gamma_2;t_\infty)Z(\Gamma_1;t_\infty)\right) > 0,$$

is invariant under $(\Gamma_1,\Gamma_2) \to (\Gamma_1,-\Gamma_2)$. Clearly the marginal-stability walls for Γ_1 with Γ_2 are anti marginal-stability walls for Γ_1 with $-\Gamma_2$, and *vice versa*.

It is natural to conjecture that at any given value of the Kähler moduli, it is impossible for both (Γ_1,Γ_2) and $(\Gamma_1,-\Gamma_2)$ to exist as two centered BPS bound states. This can be established near the boundaries of the region of Denef stability, since it is impossible for a two centered solution with charges Γ_1 and Γ_2 to exist near the wall of anti-marginal stability (and likewise the bound state Γ_1 and $-\Gamma_2$ cannot exist near the marginal stability wall of Γ_1 with Γ_2). Assuming that such a solution did exist, the separation between the centers (2.6) would diverge as t_∞ approached the anti marginal-stability wall. However the total energy of the bound state is given by

$$|Z(\Gamma_1 + \Gamma_2;t_\infty)| \to |\,|Z(\Gamma_1;t_\infty)| - |Z(\Gamma_2;t_\infty)|\,| < |Z(\Gamma_1;t_\infty)| + |Z(\Gamma_2;t_\infty)|, \quad (2.9)$$

so energy conservation would be violated in the decay of this state.

Moreover, when $|Z(\Gamma_1;t_\infty)| \gg |Z(\Gamma_2;t_\infty)|$ we have $\alpha_\infty = \arg(Z(\Gamma_1 + \Gamma_2;t_\infty) \approx \arg(Z(\Gamma_1;t_\infty)$, even away from the walls. It then follows that if we replace $(\Gamma_1,\ \Gamma_2) \to (\Gamma_1,\ -\Gamma_2)$, there will be little change in α_∞, yet the "fragment" Γ_2 has been replaced by its anti-particle. We have just argued that such a two-body solution cannot exist near the marginal stability wall for (Γ_1,Γ_2). Therefore, either something goes wrong with the supergravity solution based on (2.5), or there are new walls which "censor" such solutions from existing near the marginal-stability wall of (Γ_1,Γ_2), or there are interesting re-arrangements of the boundstate configurations as t_∞ moves towards the anti-marginal wall. (In this third possibility what we have in mind is that the boundstate should change - without a change of index - to a boundstate of $(\Gamma_1 - (n+1)\Gamma_2)$ and $n\Gamma_2$ for some positive n.) It would be interesting to understand better what really happens. [1]

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[1]We thank F. Denef for helpful remarks suggesting ways in which the supergravity solution could become singular.

3. Review of calculations of D6/D2/D0 partition functions by toric techniques

3.1 "Large radius" Donaldson-Thomas theory on toric Calabi-Yau 3-folds

The microscopic worldvolume theory of a D6 is the $6+1$ dimensional DBI action. When the brane is wrapped on a curved Calabi-Yau manifold, the supersymmetric index can be calculated in the topologically twisted version of the low energy limit of this theory. This is a 6 dimensional Euclidean topological theory obtained by twisting the $\mathcal{N} = 2$ supersymmetric $U(1)$ Yang-Mills theory in 6 dimensions.

There are singular instantons in this topological gauge theory which carry nonvanishing D2 charge $[F \wedge F] \in H^4(X; \mathbb{Z})$, and D0 charge $[F \wedge F \wedge F] \in H^6(X; \mathbb{Z})$, where F is the field strength of the D6 worldvolume abelian gauge field. These singular instantons are studied in [17], and are shown to correspond to ideal sheaves on the Calabi-Yau manifold. (See, however, the comments in section 4.2 below.)

In [17, 20] the Donaldson-Thomas partition function on the resolved conifold was computed using equivariant localization on the moduli space of ideal sheaves, building on the work of [1, 24]. The result is

$$Z_{\text{DT}}(u, v) = M(-u)^2 \prod_{j>0}(1 - (-u)^j v)^j, \tag{3.1}$$

where the McMahon function is defined by

$$M(-u) := \prod_{j>0} \frac{1}{(1 - (-u)^j)^j}. \tag{3.2}$$

This is a special case of the relation between the Donaldson-Thomas partition function and the topological string partition function on any Calabi-Yau 3-fold, expressed in terms of the BPS invariants (a.k.a. the genus zero Gopakumar-Vafa invariants),

$$Z_{\text{DT}} = \prod_{\beta_h, n_h} \left(1 - (-u)^{n_h} v^{\beta_h}\right)^{n_h n_{\beta_h}^0}. \tag{3.3}$$

We shall see that this is the partition function of D6/D2/D0 bound states only in a certain limit of the asymptotic Kähler moduli. As in [9], this limit requires a specific tuning of the B-field, even at large volume. We will see more examples of the need for such tuning below.

3.2 Szendrői's calculation of DT invariants on the noncommutative conifold

The partition function of Donaldson-Thomas theory on a noncommutative deformation of the conifold was studied in [30]. By definition, the ideal sheaves on this noncommuative space are cyclic representations of the noncommutative conifold algebra,

$$\mathcal{A} = \mathbb{C}[f_0, f_1]\langle A_1, A_2, B_1, B_2 \rangle / \langle B_1 A_i B_2 - B_2 A_i B_1, A_1 B_i A_2 - A_2 B_i A_1, i = 1, 2\rangle,$$

which is a resolution of the singular conifold algebra, $\text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 x_2 - x_3 x_4))$. The meaning of the angular brackets is that products of letters A_i and B_j which do not

form paths in the quiver, such as $A_1 A_2$, are set to zero in the algebra. The elements f_0 and f_1 are idempotent, with $f_0 f_1 = 0$, and are associated to length zero paths based at the two nodes.

There is again a torus action on the moduli space of ideal sheaves, which was exploited to reduce the calculation of the Euler character to the fixed points. These were in one to one correspondence with pyramid partitions in a length 1 empty room configuration. Using combinatoric methods, the generating function was determined to be [30, 34]

$$Z_{\text{Sz}} = M(-u)^2 \prod_{j>0} (1 - (-u)^j v)^j (1 - (-u)^j v^{-1})^j.$$

We shall see later that this is exactly the partition function of D6/D2/D0 BPS states in a particular chamber of the Kähler moduli space. Certainly a nonzero B-field is related to noncommutative gauge theory [27, 26], so it is not unexpected that turning on a ("large") B-field should produce this result. It would be worthwhile understanding in more detail why the particular value we find is the appropriate one.

4. Extended Kähler moduli space for local Calabi-Yau

In this section we motivate our extension of the Kähler moduli space of the local geometry by a single real parameter. Physically, this parameter measures the strength of a component of the B-field normal to the local $\mathbb{P}^1$.

To make sense of the brane charges in the local setting, which is crucial for being able to apply wall crossing formulae, we will consider embedding the local geometry into a compact Calabi-Yau. We proceed to determine the behavior of the central charges of various D-branes in the local limit, in which, informally, all Kähler parameters are taken to be large, with the exception of the size of the rigid curve under investigation. There is a considerable simplification of the dependence of the index of BPS D6/D2/D0 states on the Kähler moduli in that limit, analogous to that found in the large volume limit [9]. We will find that, in addition to the complexified Kähler parameter of the $\mathbb{P}^1$, an extra real parameter remains in the formulae for marginal stability walls and BPS indices, even after we take the decompactification limit.

To motivate further the extension of the Kähler moduli space of the local curve, we will show that it emerges naturally from the worldvolume perspective as well. This worldvolume theory can be made well-defined in the case of noncompact Calabi-Yau manifolds by using techniques of toric localization in the fiber. We shall see that this leads, in a somewhat different way, to the same picture of an extension of the local Kähler moduli space by an additional real parameter.

4.1 Motivation from taking a limit of compact CY

Our main example will be the resolved conifold. However, let us begin with a more general setting. We consider the local limit of a compact Calabi-Yau 3-fold, X, in which the only homology class which remains small is a rigid rational curve, dual to $\beta \in H^4(X)$. The Kähler parameter is

$$t = z\mathcal{P} + \Lambda e^{i\varphi} \mathcal{P}', \tag{4.1}$$

where $\mathcal{P}\beta = 1$, $\mathcal{P}'\beta = 0$, and Λ is a positive real number. We are interested in the behavior in the $\Lambda \to +\infty$ limit. We assume that the positive class $\mathcal{P} \in H^2(X; \mathbb{R})$ and semi-positive class $\mathcal{P}' \in H^2(X; \mathbb{R})$ are such that t is in the Kähler cone for all positive Λ. We also assume $(\mathcal{P}')^3 > 0$ (and hence $\mathcal{P}'^2 \neq 0$). [2] Note that in these variables t lies in the Kähler cone for $\varphi \in (0, \pi)$ and $\mathrm{Im}(z) > 0$.

The central charges in the local limit are easily obtained from the large volume expression for the periods. Consider a charge

$$\Gamma_1 = 1 - m'\beta + n'dV.$$

Then its central charge is given in the local limit by

$$Z(\Gamma_1; t) = \Lambda^3 e^{3i\varphi} - m'z - n' \longrightarrow \Lambda^3 e^{3i\varphi}.$$

The multi-centered solutions which can contribute to the index of states with charge Γ_1 have a core $\Gamma_{core} = 1$, together with a "halo" of particles or "fragments" of charge $\Gamma_h = -m_h\beta + n_h dV$. The latter have central charges given by

$$Z(\Gamma_h; t) = -m_h z - n_h.$$

Therefore we see that the position of the walls of marginal stability for charges of the above form only depends on the coordinates $z \in \mathbb{C}$ and $\varphi \in (0, \pi)$. The extra data needed to define the index in our noncompact setting is the variable φ, which encodes, in the local limit, all of the dependence of the central charges on the Kähler moduli introduced in the compactification. By definition, the B-field is given by $B = \mathrm{Re}(z)\mathcal{P} + \Lambda \cos \varphi \mathcal{P}'$. In the noncompact limit, it is more meaningful to talk about the local density of the B-field along those directions normal to the local curve, normalized with respect to the local value of the Kähler form. This is exactly $\cot \varphi$.

4.2 Motivation from worldvolume instanton equation

We have seen by embedding the local geometries we wish to study into a compact Calabi-Yau manifold that the D6/D2/D0 partition function depends on an additional parameter, φ, which comes from the Kähler structure in the noncompact directions. It would be gratifying to understand how this occurs from the worldvolume point of view. This will also lead us to an intrinsic definition of the D6/D2/D0 partition function as a function of the background moduli in a local Calabi-Yau 3-fold, without the need for a global completion.

Note that the central charge of the D6 brane cannot be calculated directly in the local geometry, due to its divergent volume; this is the reason we regulated by compactifying above. However, the argument of the central charge does have a meaning in this context, and this is all one needs to find the walls of marginal stability. To see this, we go back to the original worldvolume gauge theory description of D6/D2/D0 bound states.

Following the work of [17] we expect that for large Kähler class our bound states can be described as singular instantons of the $U(1)$ twisted $\mathcal{N} = 2$ gauge theory living on the

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[2]This excludes the possibility that the surface dual to $\mathcal{P}'$ is a K3 fiber.

6-brane worldvolume. They are given by singular solutions of the hermitian Yang-Mills equations,

$$F^{2,0} = 0, \qquad F^{1,1} \wedge J^2 = \ell J^3, \qquad (4.2)$$

which can be made well-defined by turning on a noncommutative deformation. In [17] it was argued that counting solutions to (4.2) is equivalent to counting ideal sheaves and hence amounts to enumeration of standard Donaldson-Thomas invariants.

However, there is a problem with this picture: It does not account for walls of marginal stability extending to infinity. [3] Thus the picture can be at best correct in certain chambers of the complexified Kähler cone. (These chambers will be located in Section 5.) One might suspect that one should apply - on the algebro-geometric side - some stability conditions, generalizing the DUY theorem that slope-stable holomorphic vector bundles are in correspondence with solutions to the Hermitian Yang-Mills equations [31, 11]. But slope stability is trivial for ideal sheaves, and correspondingly, the solutions to (4.2) do not depend on background moduli.

Of course, we should in fact be applying Π stability (a.k.a. Bridgeland stability) to elements of the derived category of coherent sheaves of charge $1 - m\beta + ndV$ [12].[4] There should be a generalization of the DUY theorem in which stable objects in the derived category (at large Kähler class) are in 1-1 correspondence with solutions of some differential equations generalizing the Hermitian Yang-Mills equation. The natural expectation from physics is that those should be (at large Kähler class) the full non-linear instanton equations, with non-vanishing B-field, found in [19]. (We may still neglect nonperturbative worldsheet instanton corrections, since these are subleading in the local limit.) The equations from [19] can be written in the form

$$\mathrm{Re}\left[e^{F+t}\sqrt{Td(X)} \right]^{(6)} = \cot 3\varphi \, \mathrm{Im}\left[e^{F+t}\sqrt{Td(X)} \right]^{(6)}, \qquad (4.3)$$

where we take the 6-form components, and recall that $t = B + iJ$. In a compact Calabi-Yau manifold, this equation would be integrated to determine the number $\cot(3\varphi)$. For local Calabi-Yau 3-folds, this integration is not well-defined, and φ becomes a free parameter. We expect that the Euler character of the moduli space of solutions depends on φ. Given the formula for the central charge, if we compactified the local geometry in some way, 3φ would become the argument of the central charge of the 6-brane, so this is the same φ we introduced above. It would be very interesting to confirm that the torus equivariant calculation of the Euler character of the moduli space of such instantons in the noncompact resolved conifold agrees with the index of BPS states obtained from the local limit of a compact Calabi-Yau manifold.

Note that the solutions to the non-linear equations are also singular, for the trivial reason that a smooth configuration with no 4-brane charge must satisfy $ch_1 = 0$ so that $F = dA$ if F is smooth, and hence $[F \wedge F]$ and $[F \wedge F \wedge F]$ must also vanish. However, the D2 and D0 charges carried by this singular solution are finite (which is not the case

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[3]A similar remark applies to the partition function counting D4/D2/D0 boundstates on Calabi-Yau manifolds with $h^{1,1}(X) > 1$ [10, 5, 2].

[4]An alternative stability condition is that studied in [25].

for the linearized equations) [26]. Thus it might be possible to analyze the moduli space of instantons for the non-linear equations without making a noncommutative deformation.

In principle, one should be able to see explicitly the appearance and disappearance of solutions (with such mild singularities) of equation (4.3) as one varies the local Kähler modulus, z as well as φ. We have not attempted to do this so we content ourselves with finding the walls where the central charges are aligned, and using the supergravity inspired wall crossing formulae of [9]. This will turn out to reproduce exactly the answer computed at the conifold point by counting equivariant quiver representations by [30, 34, 35].

5. Walls and their crossing

5.1 Which are the relevant walls?

In the local geometry, only the pure D6 brane exists as a single centered object, its attractor flow hitting a zero at the conifold point. Note that using the large volume limit of the periods, this attractor flow would hit the boundary of the Kähler moduli space at the point $z = 0$, but the periods receive significant instanton corrections in that regime, modifying the result. The existence of this BPS state is established by microscopic reasoning, as is the index of D2/D0 particles. The single centered attractor flows for other charges of the form $1 - m\beta + ndV$ always end on a zero at a regular point in the moduli space. The fact that there are no other single centered solutions carrying one unit of D6 charge (and potentially various D2 and D0 charges) is due to the vanishing of all higher genus Gopakumar-Vafa invariants in the resolved conifold.

The partition function of D6/D2/D0 bound states will receive contributions from a variety of single and multi-centered bound states. Our strategy for computing the index as a function of the background Kähler moduli will be to start with a known result, and determine the partition function in other chambers using wall crossing formulae. In particular, the higher genus Gopakumar-Vafa invariants vanish in the resolved conifold, hence there is a unique single centered object carrying one unit of D6 charge, namely the pure D6 brane itself. We will begin our analysis in the core region of the moduli space, where all multi-centered solutions are unstable, and $Z_{D6D2D0} = 1$.

First we will determine which split attractor flows can decay in the region of moduli space we are interested in. The local limit imposes strong constraints on the central charges, and we will find that only a particular class of fragments can become marginally stable. This will lead to a halo picture of the D6/D2/D0 bound states, similar to the description of the same system at large volume on one parameter Calabi-Yau manifolds in [9].

Suppose there is a split attractor flow for $\Gamma_1 + \Gamma_2 = \Gamma = 1 - m\beta + ndV$. Let $\Gamma_1 = a + \mathcal{D} - \beta_h + n_h dV$, where $a \in H^0(X; \mathbb{R})$, $\mathcal{D} \in H^2(X; \mathbb{R})$, $\beta_h \cdot \mathcal{P} = m_h$ and $\beta_h \cdot \mathcal{P}' = M_h$. The wall of marginal stability for this flow in the local limit is located where the central charges

$$Z_1 = \frac{a}{6}\Lambda^3 e^{3i\varphi} + \frac{\mathcal{D}}{2} \cdot (\Lambda e^{i\varphi}\mathcal{P}' + z\mathcal{P})^2 - M_h\Lambda e^{i\varphi} - m_h z - n_h,$$

and

$$Z_2 = \frac{1}{6}\Lambda^3 e^{3i\varphi} - mz - n - Z_1,$$

are aligned. In the $\Lambda \to \infty$ limit, these expressions simplify, and we keep only the leading terms.

It is obvious that no walls with $a > 1$ (or $a < 0$) can extend into the local regime, since then the D6 charges of the two fragments would have opposite sign, and Z_1 would be anti-aligned with Z_2, those being the dominant terms. Hence the quantization of charge implies that either $a = 0$ or $a = 1$. At the expense of reversing the roles of Γ_1 and Γ_2, we can assume that $a = 0$.

Generically, if $\mathcal{D} \neq 0$ then $\mathcal{D} \cdot \mathcal{P}'^2$ will be non-vanishing, and in the local limit, $Z_1 = \frac{1}{2}\mathcal{D} \cdot \mathcal{P}'^2 \Lambda^2 e^{2i\varphi}$, and $Z_2 = \frac{1}{6}\Lambda^3 e^{3i\varphi}$. These are never aligned inside the Kähler cone. Moreover, by assumption, $\mathcal{P}'$ is semipositive and $\mathcal{P}'^2$ is non-vanishing, so $\mathcal{P}'^2$ is dual to an element of the Mori cone (i.e. a positive sum of effective curve classes). Thus if $\mathcal{P}'^2 \mathcal{D} = 0$ then $\mathcal{D}$ is not very ample, and cannot support a single centered bound state. Since in this limit we have already shown that $D6$-$\overline{D6}$ bound states are unstable, it appears that such fragments do not exist in the regime of interest. Thus we conclude that $\mathcal{D} = 0$.

Furthermore, for $M_h \neq 0$, the associated term would be the dominant one, and the wall of marginal stability would be located at $e^{2i\varphi} \parallel -M_h$, which extends into the Kähler cone in the local limit only along the line $\varphi = \pi/2$ (if $M_h > 0$). We will see later that our entire analysis, beginning in the core region, can be applied for $\varphi < \pi/2$, so we will not cross these walls if they are present.

The physical picture that emerges from the above arguments is that in the limit of the background Kähler moduli that we are considering, the attraction between D6 branes and all other charges, with the exception of $-m_h\beta + n_h dV$, is very large, and any such BPS multi-centered solutions have radii that decreases with a power of Λ. Thus only D2/D0 fragments may become unbound as the Kähler moduli are dialed, with a resulting jump in the index of BPS states.

Therefore, all of the splits in the attractor flow tree involve a charge of the form $\Gamma_h = -m_h\beta + n_h dV$. Moreover, as we argued earlier, the only single centered configuration with D6 charge is the pure D6,[5] so it must be one of the ends of the split attractor tree. The BPS mass of the pure D6 is much greater than that of the D2/D0 fragments, thus the bound states we are considering look like "atoms" with D2/D0 particles in halos of various radii around the pure D6 core. This is the special case of the picture found in [9] in a similar context, in which only a single core state exists in the spectrum.

We have shown that the only walls of marginal stability that extend into the region of large Λ are those for the charges $1 - m'\beta + n'dV$ and $-m_h\beta + n_h dV$. Given the expressions for their central charge, the location of such walls only depends on the parameters z and φ in the $\Lambda \to \infty$ limit. Therefore, assuming the conjecture that the index of BPS bound states can only change across walls of marginal stability, we can conclude that the index

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[5]This is true for the resolved conifold geometry itself, due to the absence of high genus GV invariants. Any compact Calabi-Yau manifold in which the local curve is embedded will have a nontrivial spectrum of core states. However our argument above shows that there are no jumps in the index of those core states, so we can consider the compactification independent ratio of the D6/D2/D0 partition function in the various chambers we identify in the local limit to this Z_{core}. This is what we calculate, and identify as the partition function on the resolved conifold.

of BPS D6/D2/D0 bound states has a well defined limit, depending only on z and φ:[6]

$$\lim_{\Lambda \to \infty} \Omega_X(1 - m\beta + ndV; t) := \Omega_{\text{local}}(1 - m\beta + ndV; z, \varphi). \tag{5.1}$$

5.2 Location of the walls of marginal stability

We wish to determine the location of walls of marginal stability for $\Gamma_1 = 1 - m'\beta + n'dV$ and $\Gamma_2 = -m\beta + ndV$ in the effective Kähler moduli space we discussed in section 4. Using the above formulae for the central charges, we see that they are aligned when

$$\varphi = \frac{1}{3}\arg(-mz - n) + \frac{2\pi}{3}k,$$

for $k \in \mathbb{Z}$. These are to be thought of as real codimension 1 walls in the three real dimensional space defined by z and φ. Note that the walls are independent of the charges n' and m' in the local limit, as their contribution is subleading to that of the D6. There will be different cases for the four possible signs of m and n. For future reference, we will denote the walls of marginal stability as

$$\mathcal{W}_n^m = \{(z, \varphi) : \varphi = \frac{1}{3}\arg(z + n/m) + \frac{\pi}{3}\} \tag{5.2}$$

$$\mathcal{W}_n^{-m} = \{(z, \varphi) : \varphi = \frac{1}{3}\arg(z - n/m)\} \tag{5.3}$$

$$\tilde{\mathcal{W}}_n^{-m} = \{(z, \varphi) : \varphi = \frac{1}{3}\arg(z - n/m) + \frac{2\pi}{3}\}, \tag{5.4}$$

where $m > 0$. The structure of the walls turns out to be most clearly visible by fixing z and varying φ in its allowed range from 0 to π.

To connect this three dimensional moduli space with the one Kähler parameter analysis of [9], it is useful to consider a special class of Kähler forms given by $t = xJ_0$, where $x \in \mathbb{C}$ and J_0 is real. Such geometries have B proportional to J as vectors in $H^2(X; \mathbb{R})$, and the periods naturally correspond to those found in the one parameter case. In our variables, this condition implies that $\varphi = \arg(z)$.

Therefore we can identify the core region, which is the chamber containing the limit of large volume and zero B-field, where in our case only the pure D6 exists (due to the absence of higher genus GV invariants in the resolved conifold), as an open region around $\varphi = \pi/2$, $z = iy$ for sufficiently large positive y, following the results of [9]. There are no relevant walls between this point and $\varphi = \frac{1}{3}\arg(z) + \frac{\pi}{3}$ (for any value of z in the upper half plane) , so we shall start our analysis there, where $Z_{D6/D2/D0} = 1$.

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[6]It is important that the local curve is rigid, otherwise the degeneracies of the D2/D0 fragments themselves (the Gopakumar-Vafa invariants) in the compact Calabi-Yau manifold, X, would differ from their values in the equivariant regulation of the noncompact local Calabi-Yau manifold. In the case of the resolution of local $A_1 \times \mathbb{C}$, this difficulty can be overcome by partially compactifying to toric local rigid surface, such as $\mathbb{P}^1 \times \mathbb{P}^1$, and then embedding in a compact Calabi-Yau manifold.
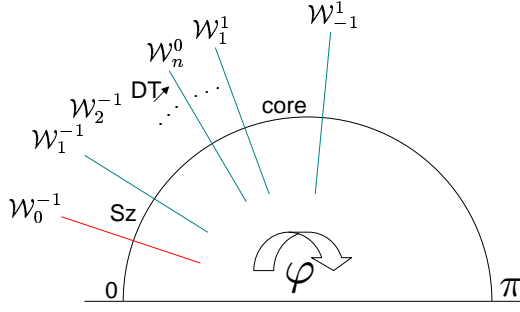
Figure 1: This figure illustrates the position of the walls of marginal stability at fixed z. The core region, Donaldson-Thomas limit, and Szendrői region are labelled.

5.3 Computing the partition function in all the chambers

In the resolved conifold, the BPS indices are known, and the degeneracies of primitive D2/D0 particles can be easily extracted. In particular, the only non-vanishing ones are [15, 16]

$$\Omega(\pm\beta + ndV) = 1 \tag{5.5}$$

valid for all n, and

$$\Omega(ndV) = -2, \tag{5.6}$$

valid for $n \neq 0$, and these degeneracies have no walls of marginal stability at large radius by arguments similar to those used in section 5.1. The D2/D0 fragments give a net fermionic contribution to the index, and their contribution to the D6/D2/D0 partition function will be identical to that of particles obeying Fermi-Dirac statistics, while the index of the D0 fragments is net bosonic, so their contribution to the partition function will be identical to that of particles obeying Bose-Einstein statistics.

Thus we only need to consider the walls $\mathcal{W}_n^{\pm 1}$, $\mathcal{W}_n^0$, which are simple enough to draw; see figure 1. The chambers between successive walls will be denoted as $[\mathcal{W}_{n_1}^{m_1}\mathcal{W}_{n_2}^{m_2}]$ where we assume $\mathcal{W}_{n_1}^{m_1}$ is to the left of $\mathcal{W}_{n_2}^{m_2}$ in the φ direction. Starting in the core region, as we decrease φ, increasing the B-field in the directions normal to the local curve, the collection of walls for Γ_1 with $-m\beta + ndV$ with $n, m > 0$ are crossed first. They are located at $\varphi = \frac{1}{3}\arg(z - |n/m|) + \pi/3$. Note that $0 < \frac{1}{3}\arg(z + |n/m|) < \frac{1}{3}\arg(z - |n/m|) < \pi/3$, so these are indeed the first walls we cross. A necessary condition for the bound state $\Gamma_1 + \Gamma_2$ to exist is the Denef stability criterion:

$$\langle \Gamma_1, \Gamma_2 \rangle \text{Im}(Z_1 Z_2^*) = -n\text{Im}(e^{3i\varphi}(-mz^* - n)) > 0. \tag{5.7}$$

Clearly this is satisfied when φ is less than $\frac{1}{3}\arg(z - |n/m|) + \pi/3$. A closer examination of the supergravity solution reveals no pathologies and we conclude that the bound state enters the spectrum as φ crosses the wall in the direction of smaller values. There are no nontrivial core states in this local model. Therefore, in the chamber $[\mathcal{W}_{n+1}^1\mathcal{W}_n^1]$, the

partition function is given by

$$Z_n^+(u,v) = \prod_{j=1}^{n}(1-(-u)^j v)^j, \tag{5.8}$$

in accord with the semiprimitive wall crossing formula (2.8).

There are an infinite number of walls $\mathcal{W}_n^1$, one for each $n > 0$, however the index for any given total charge remains finite as $n \to \infty$, since the successive fragments have ever increasing D0 charge, and eventually do not contribute to the index for a fixed total charge. These walls have an accumulation point at $\varphi = \pi/3$, which is the location of the wall of marginal stability of D6 with pure D0. The limit from the right,

$$\lim_{n\to\infty} Z_n^+(u,v) = \prod_{k>0}(1-(-u)^k v)^k = Z'_{\mathrm{DT}}(u,v) \tag{5.9}$$

is the reduced Donaldson-Thomas partition function defined in [20].

Continuing to increase the B-field in the normal directions, the central charges of the D6 bound state aligns with that of $\overline{D2}/D0$ fragments when $\varphi = \frac{1}{3}\arg(z-|n/m|)$ for $m < 0$, $n > 0$. In the chamber $[\mathcal{W}_n^{-1}\mathcal{W}_{n+1}^{-1}]$, from the semiprimitive wall crossing formula (2.8) we have the D6/D2/D0 generating function

$$Z_n^-(u,v) = M(-u)^2 \prod_{k>0}(1-(-u)^k v)^k \prod_{m>n}(1-(-u)^m v^{-1})^m. \tag{5.10}$$

The factor of $M(-u)^2$ appears when crossing the D6D0 wall at $\varphi = \pi/3$. Note that the walls $\mathcal{W}_n^{-1}$ also accumulate at $\varphi = \pi/3$ for $n \to \infty$, so that the limit from the left is

$$\lim_{n\to\infty} Z_n^-(u,v) = M(-u)^2 \prod_{k>0}(1-(-u)^k v)^k = Z_{\mathrm{DT}}(u,v), \tag{5.11}$$

where Z_{DT} is given in equation (3.1).

Continuing to cross all the walls $\mathcal{W}_n^{-1}$ to smaller values of φ we enter the region, $\frac{1}{3}\arg z < \varphi < \frac{1}{3}\arg(z-1)$ where the partition function is given by the Szendrői form,

$$Z_{\mathrm{Sz}} = M(-u)^2 \prod_{k>0}(1-(-u)^k v)^k(1-(-u)^k v^{-1})^k. \tag{5.12}$$

We will refer to this as the Szendrői region.

The conifold point lies in the boundary of the Szendrői region as $\mathrm{Im}(z)$ goes to zero. Note that (5.12) is precisely the partition function computed by Szendrői by counting quiver representations at the conifold point. Continuing to $\varphi < \frac{1}{3}\arg z$ leads to a breakdown in the supergravity approximation, and results in puzzles whose resolution is beyond the scope of this paper. We will discuss these puzzles in the next section.

It is natural to speculate that an analogous behavior would be observed in the D6/D2/D0 on other local curves. In particular, the analysis of the positions of the walls of marginal stability in the effective three real dimensional moduli space we defined did not depend on the triple intersection numbers of the local Calabi-Yau manifold. Thus we are motivated

to conjecture that in the analog of the Szendrői region (some neighborhood on one side of the codimension one wall $\varphi = \frac{1}{3}\arg z$), the D6/D2/D0 partition function will have the form

$$Z(u,v) = M(-u)^\chi \prod_{\beta_h, n_h} \left(1 - (-u)^{n_h} v^{\beta_h}\right)^{n_h n_{\beta_h}^0} \left(1 - (-u)^{n_h} v^{-\beta_h}\right)^{n_h n_{\beta_h}^0}, \qquad (5.13)$$

in terms of the Gopakumar-Vafa invariants. This structure has been observed in local A_1 in [35] using techniques similar to [30] at the orbifold point.

6. The D6-D2 threshold stability wall, and some puzzles

In this section we discuss some puzzles that arise when we consider the supergravity approach to wall-crossing in the vicinity of $\varphi = \frac{1}{3}\arg z$. On this wall the central charges of the D6 and $\overline{D2}$ branes become aligned. Nevertheless, the index is not expected to change since these charges are mutually local: $\langle 1, \beta \rangle = 0$. Physically, there are no bound states of a D6 and D2, so there is no halo whose radius diverges. This locus of alignment of the central charges of mutually local BPS states is called a "threshold stability wall" [5]. (See also [3] for a discussion of such walls.)

One might consider continuing the analysis of the partition function across the walls $\mathcal{W}_{-n}^{-1}$ at $\varphi = \frac{1}{3}\arg(z+n)$ for $n > 0$. These walls have an accumulation point at $\varphi = 0$. When decreasing φ, one would be passing from the region of Denef stability to a region of instability. This immediately presents a problem, since no bound states with net negative D0 charge are present in the spectrum we found in the Szendrői region.

Now, recall the discussion centered on (2.9). It follows from those remarks that there cannot be any halo solutions containing $-\beta + ndV$ fragments for $n > 0$ orbiting a D6 core as the walls $\mathcal{W}_{-n}^{-1}$ are approached. This suggests that the halo picture of the bound states must break down in the region around $\varphi = \frac{1}{3}\arg z$. We shall see that indeed a number of surprising phenomena occur at precisely this value of φ.

First, we note that the halo radii associated to all of the $\pm\beta + ndV$ and ndV fragments become equal at this threshold wall, suggesting a potential mechanism for a reorganization of the spectrum contributing to the index. This will be found to be reflected in the degeneration of the split attractor flow trees for our D6/D2/D0 bound states as t_∞ approaches the wall. Next, all of the D6/D2/D0 solutions, including the pure D6 brane solution, will be shown to exit the regime of validity of supergravity at the threshold wall. The reason for this is that the spatially dependent Kähler modulus of the $\mathbb{P}^1$ crosses the boundary of the Kähler cone somewhere in the interior of the solution.

Applying (2.5) we see that the supergravity solution describing a D6 brane surrounded by D2/D0 particles in halos is described by

$$H(\vec{x}) = H_\infty + \frac{1}{|\vec{x}|} + \sum_i \frac{-\beta_i + n_i dV}{|\vec{x} - \vec{x_i}|}, \qquad (6.1)$$

where the halo radii are determined from the integrability conditions to be

$$|\vec{x_i}| = r_i = \frac{-n_i}{\text{Im}\left(e^{-3i\varphi}(m_i z + n_i)\right)} \sqrt{\frac{J_\infty^3}{3}}. \qquad (6.2)$$

This simple result is valid because all of the fragments are mutually local, and hence produce no forces on each other. The positivity of the above radii is exactly the Denef stability condition. Note that this does not distinguish whether a given D2/D0 particle or its anti-particle (related by n_i, $m_i \to -n_i$, $-m_i$) will be present in a supersymmetric bound state. Let ρ denote the radius of the pure D0 halos. These are obtained by putting $m_i = 0$ to give

$$\rho = \frac{1}{\sin 3\varphi} \sqrt{\frac{J_\infty^3}{3}}. \tag{6.3}$$

In terms of ρ we can write

$$r_i = \rho \frac{\sin 3\varphi}{\sin 3\varphi + \frac{m_h}{n_h} \mathrm{Im}(e^{3i\varphi} z^*)}. \tag{6.4}$$

At the threshold stability wall of D6 and $\overline{D2}$ we see that the radii of all halos coincide, since by (6.4) $r_i = \rho$ when $\arg z = 3\varphi$. Different components of the moduli space of supergravity solutions with equal total charge become connected at this location. For example, consider the bound state, I, of $\Gamma_{core} = 1$, $\Gamma_1 = -\beta + dV$, and $\Gamma_2 = +\beta + dV$, and another bound state, II, of $\Gamma_{core} = 1$ and $\Gamma_3 = 2dV$. These contribute to the same index. In the Szendrői region, for $\varphi > \frac{1}{3}\arg z$, the fragments orbit in halos of different radii that scale like $\Lambda^{\frac{3}{2}}$, so these two multi-centered solutions are disconnected. At the threshold wall, the radii are equal, and the subspace of the moduli space of configurations of type I where the fragments Γ_1 and Γ_2 have the same angular position is indistinguishable from the moduli space of configurations of type II.

A similar degeneration can be observed in the split attractor flow trees for our D6/D2/D0 bound states. Consider the split attractor flow for a total charge $\Gamma = 1 - m\beta + ndV$ into $\Gamma_h = -m_h\beta + n_h dV$ and $\Gamma - \Gamma_h$. The attractor flow of Γ, beginning at t_∞, is described by the attractor flow equation

$$2e^{-U}\mathrm{Im}(e^{-i\alpha}\Omega_{\mathrm{nrm}}) = 2\mathrm{Im}(e^{-i\alpha_\infty}\Omega_{\mathrm{nrm}}|_\infty) - \Gamma\tau, \tag{6.5}$$

where the parameter τ is related to the spatial position by $\tau = 1/r$. The branching point of the tree is located along the flow at $\tau = \frac{1}{r_h}$, where the split occurs. At the threshold wall, we saw that (for $\Lambda \to \infty$) all r_h are equal, independent of m_h and n_h, thus the branch points for any such D2/D0 fragment are coincident. But by definition, the split is located at the intersection of the attractor flow for Γ with the wall of marginal stability for Γ and Γ_h.

Thus when t_∞ is located on the D6-$\overline{D2}$ wall, the branch point of the split attractor flow tree is located at the intersection of the walls of marginal stability of Γ with all charges of the form $-m_h\beta + n_h dV$. However it is impossible for the central charges $Z(-m_h\beta + n_h dV; t) = -m_h z - n_h$ for be aligned for all values of m_h and n_h. Therefore the branch point of the split flow must be hitting the zero of the exact function $Z(\Gamma; t)$. In general, this zero is located at small volume, which suggests that the supergravity approximation is breaking down, assuming that the graph of the split attractor flow in the Kähler moduli space is a subset of the full range of moduli in the associated supergravity solution.

To establish fully the existence of a supergravity solution, one must check that the local Kähler moduli remain in the Kähler cone, and that the local discriminant is everywhere positive. We can parameterize the harmonic function governing the supergravity solution in the form

$$H(\vec{x}) = re^{S}\left(1 - Y^2 + NdV\right),\tag{6.6}$$

where $S \in H^2(X;\mathbb{R})$, while $Y \in H^2(Y;\mathbb{R})$ is in the Kähler cone, and $N \in \mathbb{R}$.

The standard attractor solution formulae [28, 21, 4, 5] allow us to express the local moduli in terms of the components of the harmonic function $H(\vec{x})$. We begin by solving

$$D_{ABC}y^A y^B = -2H_C H^0 + D_{ABC}H^A H^B,\tag{6.7}$$

where D_{ABC} are the triple intersection numbers for y^A in the Kähler cone. Then we form $Q^3 = \left(\frac{1}{3}D_{ABC}y^A y^B y^C\right)^2$ and

$$\Sigma = \sqrt{\frac{Q^3 - L^2}{(H^0)^2}},$$

where $L = H_0(H^0)^2 + \frac{1}{3}D_{ABC}H^A H^B H^C - H^A H_A H^0$. Finally, we have

$$\mathrm{Im}(t^A) = y^A \frac{\Sigma}{Q^{\frac{3}{2}}}.\tag{6.8}$$

Consider the behavior of the Kähler moduli at the radius of the D0 halo, $|\vec{x}| = \rho$. For any value of the background moduli, H^0 vanishes along the sphere with this radius. Therefore (6.7) implies that

$$y^A = -H^A = -\sqrt{\frac{3}{J_\infty^3}}\mathrm{Im}(e^{-3i\varphi}(z\mathcal{P}^A + \Lambda e^{i\varphi}(\mathcal{P}')^A)),\tag{6.9}$$

where we use the fact that no D4 charge is present, so the H^A are spatially constant. Here we have chosen the branch of solutions with the minus sign so that the coefficient of $\mathcal{P}'$ is positive, thus $\mathrm{Im}(t^A) > 0$ for $\varphi > \frac{1}{3}\arg(z)$.

The asymptotic values of the moduli were chosen to be in the local limit, with the $\mathbb{P}^1$ small relative to the total volume of the Calabi-Yau manifold. Given that $\mathcal{P}'\beta = 0$, we can chose a positive basis of $H^2(X;\mathbb{R})$ such that $(\mathcal{P}')^1 = 0$ and $\mathcal{P}^1 = 1$. In that basis the Kähler parameter of the $\mathbb{P}^1$ is exactly $\mathrm{Im}(t^1)$. On the sphere $|\vec{x}| = \rho$, we then have that the local value of that modulus is

$$\mathrm{Im}(t^1) = \sqrt{\frac{3}{J_\infty^3}}\frac{\Sigma(\vec{x})}{Q^{\frac{3}{2}}(\vec{x})}Im(e^{-3i\varphi}z).\tag{6.10}$$

In the limit that $H^0 \to 0$, the discriminant remains finite, and is given by

$$\Sigma = \sqrt{2H_A H^A - \frac{H_0}{3}D_{ABC}H^A H^B H^C}.\tag{6.11}$$

On the other hand, Q is nonvanishing, as follows from (6.9). Putting everything together, we find that at the threshold wall, $\mathrm{Im}(t^1)$ actually vanishes on the sphere of radius ρ, even

in the pure D6 solution. This radius is large, scaling as $\Lambda^{3/2}$ in the local limit, so we are witnessing a clear breakdown of the supergravity approximation - the Kähler moduli exit the Kähler cone. It seems plausible that if one used the quantum corrected periods, the Calabi-Yau would be in the flopped conifold phase for $|\vec{x}| < \rho$. By continuity, as one approaches the wall $3\varphi = \arg z$ from the right, the modulus of the $\mathbb{P}^1$ begins to shrink in the interior of the solution.

It is possible that the spectrum of BPS states remains unchanged across the threshold wall, however the tools of this paper are insufficient to determine whether that is the case. Let us assume the validity of the conjecture that the index can only jump at walls of marginal stability. Suppose that one knew that partition function in the region $R = [\mathcal{W}_{-2}^{-1}\mathcal{W}_{-1}^{-1}]$ as a formal power series in u, u^{-1} and v, v^{-1}. Then an application of the semiprimitive wall crossing formula would imply that

$$Z_{\mathrm{Sz}}(u,v) = Z(u,v;R)\left(1 - (-u)^{-1}v^{-1}\right), \tag{6.12}$$

where we recall that the bound states of $1 - m'\beta + n'dV$ with $+\beta - dV$ are Denef-stable in Szendrői region, and Denef-unstable in the new region, R. Moreover we have used $\Omega(+\beta - dV) = 1$.

If we formally solve for $Z(u,v;R)$ by multiplying by the inverse power series $(1 + u^{-1}v^{-1})^{-1}$ we obtain the generating function

$$Z(u,v;R) = uvM(-u)^2 \prod_{p>1}(1 - (-u)^p v)^p \prod_{k>0}\left(1 - (-u)^k v^{-1}\right)^k. \tag{6.13}$$

This expression is rather unsatisfactory since it implies that the index of BPS states with total charge given by the pure D6 is vanishing in the chamber R. The microscopic theory of the pure D6 is the topologically twisted $U(1)$ gauge theory in six dimensions, which would seem to have a unique ground state in the sector with no instantons for any value of the Kähler moduli.

One should be wary of such formal manipulations of wall-crossing formulae. What the formula really states is that the indices of BPS states of the two sides of the wall are related by

$$\Omega(1 - m\beta + ndV; Sz) = \Omega(1 - m\beta + ndV; R) + \Omega(1 - (m+1)\beta + (n+1)dV; R). \tag{6.14}$$

It is clear that, given the partition function in the Szendrői region, the index in the region R is not uniquely determined by this wall crossing formula. This should be contrasted with the situation when one knows the BPS indices on the unstable side of a marginal stability wall, in which case the wall crossing formulae do uniquely determine the answer on the stable side. The non-uniqueness in our case can be parametrized by noting that given arbitrary $\Omega(1 + ndV; R)$, a solution to the wall crossing formula (6.14) can be found, with the degeneracies $\Omega(1 - m\beta + ndV; R)$ for $m \neq 0$ now determined.

Indeed, if we insist that $\Omega(1; R) = 1$ then we find

$$\Omega(1 + n\beta - ndV; R) = (-1)^n \text{ for } n \geq 0, \tag{6.15}$$

$$\Omega(1 - \beta + dV; R) = 0, \tag{6.16}$$

$$\Omega(1 - n\beta + ndV; R) = (-1)^n \text{ for } n \geq 2, \tag{6.17}$$

by solving the equations (6.14) recursively. This also seems problematic, at least in the supergravity picture, since the BPS index is now non-vanishing for bound states with exactly opposite fragment charges. (Recall the discussion at the end of section 2.) On the other hand, since we have exited the regime of validity of the supergravity approximation, the counting of BPS states in this region is beyond the scope of the tools of this paper. Once again, it would be extremely interesting to understand what really happens in this regime.

Acknowledgements

References

[1] M. Aganagic, A. Klemm, M. Marino and C. Vafa, "The topological vertex," Commun. Math. Phys. **254**, 425 (2005) [arXiv:hep-th/0305132].

[2] E. Andriyash and G. W. Moore, "Ample D4-D2-D0 Decay," arXiv:0806.4960 [hep-th].

[3] P. S. Aspinwall, A. Maloney and A. Simons, "Black hole entropy, marginal stability and mirror symmetry," JHEP **0707**, 034 (2007) [arXiv:hep-th/0610033].

[4] B. Bates and F. Denef, "Exact solutions for supersymmetric stationary black hole composites," [arXiv:hep-th/0304094].

[5] J. de Boer, F. Denef, S. El-Showk, I. Messamah and D. Van den Bleeken, "Black hole bound states in AdS3 x S2," arXiv:0802.2257 [hep-th].

[6] F. Denef, "Supergravity flows and D-brane stability," JHEP **0008**, 050 (2000) [arXiv:hep-th/0005049].

[7] F. Denef, "On the correspondence between D-branes and stationary supergravity solutions of type II Calabi-Yau compactifications," arXiv:hep-th/0010222.

[8] F. Denef, "Quantum quivers and Hall/hole halos," JHEP **0210**, 023 (2002) [arXiv:hep-th/0206072].

[9] F. Denef and G. W. Moore, "Split states, entropy enigmas, holes and halos," arXiv:hep-th/0702146.

[10] E. Diaconescu and G. W. Moore, "Crossing the Wall: Branes vs. Bundles," arXiv:0706.3193 [hep-th].

[11] S. Donaldson, "Infinite determinants, stable bundles and curvature," Duke Math. J. **54**, 1 (1987).

[12] M. R. Douglas, B. Fiol and C. Romelsberger, "Stability and BPS branes," JHEP **0509**, 006 (2005) [arXiv:hep-th/0002037].

[13] S. Ferrara, R. Kallosh and A. Strominger, "N=2 extremal black holes," Phys. Rev. D **52** (1995) 5412 [arXiv:hep-th/9508072].

[14] D. Gaiotto, G. W. Moore and A. Neitzke, "Four-dimensional wall-crossing via three-dimensional field theory," arXiv:0807.4723 [hep-th].

[15] R. Gopakumar and C. Vafa, "M-theory and topological strings. I," arXiv:hep-th/9809187.

[16] R. Gopakumar and C. Vafa, "M-theory and topological strings. II," arXiv:hep-th/9812127.

[17] A. Iqbal, N. Nekrasov, A. Okounkov and C. Vafa, "Quantum foam and topological strings," arXiv:hep-th/0312022.

[18] M. Kontsevich and Y. Soibelman, "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations," to appear.

[19] M. Marino, R. Minasian, G. W. Moore and A. Strominger, "Nonlinear instantons from supersymmetric p-branes," JHEP **0001**, 005 (2000) [arXiv:hep-th/9911206].

[20] D. Maulik, N. Nekrasov, A. Okounkov, R. Pandharipande, "Gromov-Witten theory and Donaldson-Thomas theory. I," Compos. Math. **142** 5 (2006) [arXiv:math/0312059v3].

[21] G. W. Moore, "Arithmetic and attractors," arXiv:hep-th/9807087.

[22] K. Nagao and H. Nakajima, "Counting invariant of perverse coherent sheaves and its wall-crossing," arXiv:0809.2992.

[23] K. Nagao, "Derived categories of small toric Calabi-Yau 3-folds and counting invariants," arXiv:0809.2994.

[24] A. Okounkov, N. Reshetikhin and C. Vafa, "Quantum Calabi-Yau and classical crystals," arXiv:hep-th/0309208.

[25] R. Pandharipande and R. P. Thomas, "Stable pairs and BPS invariants," arXiv:0711.3899 [math.AG].

[26] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909**, 032 (1999) [arXiv:hep-th/9908142].

[27] V. Schomerus, "D-branes and deformation quantization," JHEP **9906**, 030 (1999) [arXiv:hep-th/9903205].

[28] M. Shmakova, "Calabi-Yau black holes," Phys. Rev. D **56**, 540 (1997) [arXiv:hep-th/9612076].

[29] A. Strominger, "Macroscopic Entropy of $N = 2$ Extremal Black Holes," Phys. Lett. B **383** (1996) 39 [arXiv:hep-th/9602111].

[30] B. Szendrői, "Non-commutative Donaldson-Thomas invariants and the conifold," Geom. Topol. **12**, 2 (2008) [arXiv:0705.3419].

[31] K. Uhlenbeck and S.-T. Yau, "On the existence of Hermitian-Yang-Mills connections in stable vector bundles," Comm. Pure Appl. Math. **39**, no. S (1986).

[32] C. Vafa and E. Witten, "A Strong coupling test of S duality," Nucl. Phys. B **431**, 3 (1994) [arXiv:hep-th/9408074].

[33] C. Vafa, "Two dimensional Yang-Mills, black holes and topological strings," arXiv:hep-th/0406058.

[34] B. Young, "Computing a pyramid partition generating function with dimer shuffling," arXiv:0709.3079v2 [math.CO]

[35] B. Young and J. Bryan, "Generating functions for colored 3D Young diagrams and the Donaldson-Thomas invariants of orbifolds," arXiv:0802.3948 [math.CO].